SIEGEL DISKS AND PERIODIC RAYS OF ENTIRE FUNCTIONS

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Abstract. Let f be an entire function whose set of singular values is bounded and suppose that f has a Siegel disk U such that $f|_{\partial U}$ is a homeomorphism. We show that U is bounded. Using a result of Herman, we deduce that if additionally the rotation number of U is Diophantine, then ∂U contains a critical point of f.

Suppose furthermore that all singular values of f lie in the Julia set. We prove that, if f has a Siegel disk U whose boundary contains no singular values, then the condition that $f\colon \partial U \to \partial U$ is a homeomorphism is automatically satisfied. We also investigate landing properties of periodic dynamic rays by similar methods.

1. Introduction

Main Results. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonlinear entire function and suppose that U is a Siegel disk of f, i.e. an invariant component of the Fatou set on which f is conformally conjugate to an irrational rotation. It is an important question under which conditions ∂U contains a (finite) singular value of f.

Herman [H] proved the following result. *If the rotation number of U is Diophantine*[1]*, if U is bounded and if $f\colon \partial U \to \partial U$ is a homeomorphism, then ∂U contains a critical point.*

Here the condition that $f|_{\partial U}$ be a homeomorphism is needed to exclude certain topological "pathologies" (compare [Ro]); it is currently unknown whether these can actually occur. With this reasonable restriction, Herman's theorem gives a very satisfactory answer when f is a polynomial, since U is always bounded in this case. On the other hand, the result is far less complete for transcendental functions. For example, it does not answer the question whether the boundary of the Siegel disk depicted in Figure 1(a) does indeed contain the singular value. In this article, we prove that the assumption of boundedness can be removed for a large class of entire functions.

Theorem 1 (Univalent Siegel disks)**.**
Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function whose set of singular values is bounded and suppose that U is a Siegel disk of f with the property that $f\colon \partial U \to \partial U$ is a homeomorphism. Then U is bounded.

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[1]Diophantine rotation numbers can be replaced by the larger class $\mathcal{H}$ described by Yoccoz [Y]; compare the discussion in [Pé, Chapter I].

In particular, if the rotation number of U is Diophantine, then ∂U contains a critical point.

Remark. The class of entire functions whose set of singular values is bounded is called the *Eremenko-Lyubich class* and is commonly denoted by $\mathcal{B}$. This is the natural class of entire functions to which our methods apply. It seems by no means clear whether one should expect Siegel disk boundaries to contain finite singular values for more general classes of maps.

Since the condition of Theorem 1 is often difficult to verify directly, we also prove the following. (Here $J(f)$ and $S(f)$ denote the Julia set and set of singular values of f, respectively; compare the remarks on notation below.)

Theorem 2 (Nonsingular Siegel disks).
Let $f \in \mathcal{B}$ with $S(f) \subset J(f)$. If U is a Siegel disk of f which satisfies $S(f) \cap \partial U = \emptyset$, then $f\colon \partial U \to \partial U$ is a homeomorphism.

Remark. If f is an exponential map, i.e. $f(z) = e^{2\pi i\vartheta}(\exp(z) - 1)$, and f has a Siegel disk, then the unique singular value of f must automatically belong to the Julia set. So Theorems 1 and 2 imply that the boundary of an unbounded Siegel disk of an exponential map always contains the singular value, which answers a question of Herman, Baker and Rippon [BrH, Problem 2.86 (b)]. The proof for this special case has previously appeared in [R2] and was obtained independently by Buff and Fagella [BF].

Combined with Herman's result mentioned above, the preceding theorems yield the following corollary.

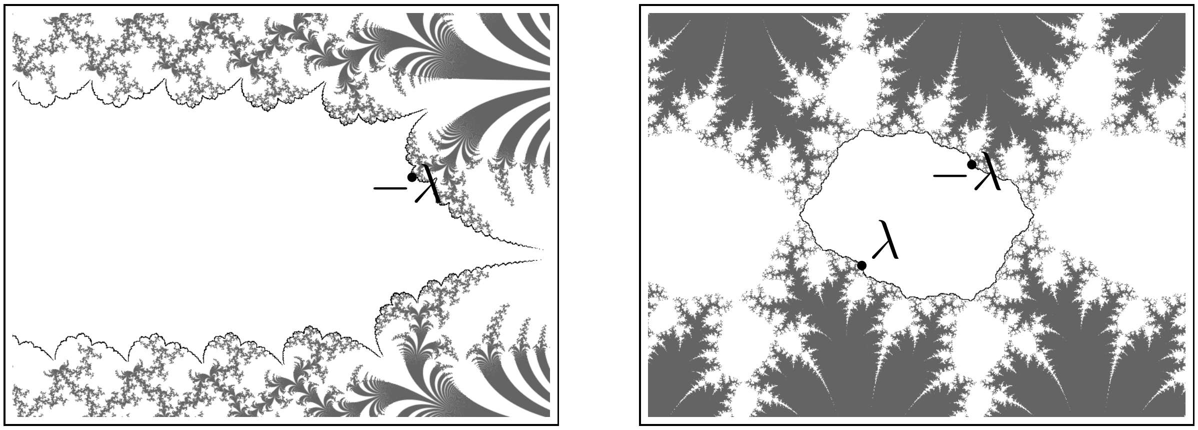
Corollary (Diophantine Siegel disks).
Let $f \in \mathcal{B}$ with $S(f) \subset J(f)$, and suppose that f has a Siegel disk U with diophantine rotation number. Then $S(f) \cap \partial U \neq \emptyset$.

Finally, if f has only two critical values and no asymptotic values, we can give a complete result on the boundedness of nonsingular Siegel disks. (An important example of such functions is given by the family $C_{a,b}\colon z \mapsto a\exp(z) + b\exp(-z)$ of cosine maps, where $a, b \in \mathbb{C} \setminus \{0\}$.)

Theorem 3 (Maps with two critical values).
Let f be an entire function which has two critical values and no asymptotic values, and suppose that f has a periodic (i.e., not necessarily invariant) Siegel disk U such that, for all $j \geq 0$, the boundary $\partial f^j(U)$ contains no critical values of f. Then U is bounded.

Remark 1. In the case where both critical values of f lie in the Julia set, it is sufficient to demand that no $\partial f^j(U)$ contains both of these.

Remark 2. It does not seem unreasonable to expect that all Siegel disks in the cosine family are bounded, but the difficulties involved in showing this are unresolved even for polynomials. For instance, it is not known whether a quadratic polynomial can have a Siegel disk whose boundary is the entire Julia set. A cosine Siegel disk whose boundary is the Julia set would in particular be unbounded.



(a) $f(z) = \lambda(\exp(z) - 1)$ (b) $f(z) = \lambda \sin(z)$

FIGURE 1. Two entire functions with a Diophantine Siegel disk (in both cases the rotation number is the golden mean). Our results show that both Siegel disk boundaries do indeed contain a singular value.

The methods which yield the above theorems apply, in fact, to a wide range of connected invariant sets. As a second example, we discuss the application to periodic rays. A *fixed ray* of an entire function f is a curve

$$\gamma \colon (-\infty, \infty) \to \mathbb{C}$$

with $\lim_{t\to+\infty} |\gamma(t)| = \infty$ which satisfies $f(\gamma(t)) = \gamma(t+1)$ for all t. As usual, we say that γ *lands* at a point $z_0 \in \hat{\mathbb{C}}$ if $\lim_{t\to-\infty} \gamma(t) = z_0$. A *periodic ray* of f is a fixed ray of some iterate f^n of f.

Periodic rays play an essential part in the study of polynomial dynamics. It is now known that such rays exist for many, although not for all, functions f in the Eremenko-Lyubich class, in particular for those of finite order [R^{3}S]. They have already been used to great advantage in the theory of exponential maps, see e.g. [SZ, S2], and are likely to be equally useful in the study of more general families.

Much of the usefulness of periodic rays for polynomials depends on the fact that every such ray lands at a repelling or parabolic periodic point (see [M, Theorem 18.10] or our Corollary B.5). While the proof of this result breaks down in the transcendental case (compare Appendix B), the author was recently able to generalize it to exponential maps [R3]. This gives us reason to expect that such a result remains true for some larger classes of entire functions. However, the proof in [R3] uses a theorem of Schleicher [S1] on exponential parameter space which depends essentially on the fact that this parameter space is one-dimensional. We will prove the following result.

Theorem 4 (Landing of periodic rays)**.**
Let $f \in \mathcal{B}$. If γ is a fixed ray of f such that $f \colon \overline{\gamma} \to \overline{\gamma}$ is a homeomorphism and the accumulation set of γ does not contain any critical points, then γ lands at a repelling or parabolic fixed point of f.

Similarly, if $S(f) \subset J(f)$ and γ is a fixed ray of f whose accumulation set does not intersect $S(f)$, then γ lands at a repelling or parabolic fixed point of f.

As far as we know, this is the first landing criterion not relying on hyperbolic expansion which can be applied to functions in higher-dimensional parameter spaces.

Idea and Structure of the Proof. As already mentioned, our results are not restricted to the special (though important) cases described above. Their basis lies in the following general principle.

Theorem 5 (Invariant connected sets)**.**
Let $f \in \mathcal{B}$ and suppose that $A \subset \mathbb{C}$ is closed and connected such that $f(A) \subset A$ and such that $f\colon A \to \overline{f(A)}$ is a homeomorphism.

Then for every $R > 0$, there exists $R' > 0$ such that

$$\{z \in A\colon |z| \geq R'\} \subset \{z \in \mathbb{C}\colon f^n(z) \geq R \text{ for all } n \text{ and } |f^n(z)| \to \infty \text{ as } n \to \infty\}.$$

This result implies Theorem 1 by letting A be the closure of the Siegel disk U and using the fact that U cannot contain escaping points. (See Section 3 for details.)

The proof of Theorem 5 is, in fact, quite simple. The hypothesis implies that the unbounded parts of A are contained in finitely many *fundamental domains* of f (for cosine maps $C_{a,b}$ as above this is equivalent to $\operatorname{Im} A$ being bounded). An expansion argument then shows that any sufficiently large point in A has a large image (this is akin to the fact that $|C_{a,b}(z)|$ behaves like $\exp(|\operatorname{Re} z|)$ when $|\operatorname{Re} z|$ is large), yielding the desired result.

This proof is carried out in Sections 2 and 3, with the former section reviewing basic definitions for functions in the Eremenko-Lyubich class and deducing the abovementioned expansion statement, and the latter containing the actual proof.

The remaining two sections show how Theorem 5 can be applied in cases where there are no singular values in the Fatou set, and how to apply our results to the landing problem for periodic rays.

Two auxiliary results of a topological nature were relegated to Appendix A to avoid interrupting the flow of ideas. Appendix B discusses difficulties in proving landing results for periodic dynamic rays using hyperbolic contraction.

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Notation. We denote the complex plane and the Riemann sphere by $\mathbb{C}$ and $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, respectively. The unit disk is denoted by $\mathbb{D}$; more generally, $\mathbb{D}_r(z_0)$ denotes the open disk of radius r around some $z_0 \in \mathbb{C}$. The unit circle is denoted by S^1.

The underlying topological space for our considerations is the complex plane $\mathbb{C}$; all closures, boundaries, neighborhoods etc. will be understood to be taken in $\mathbb{C}$ unless explicitly stated otherwise. The Euclidean length of a curve γ is denoted by $\ell(\gamma)$

Throughout this article (with the exception of the first half of Section 5), $f\colon \mathbb{C} \to \mathbb{C}$ will be a nonconstant nonlinear entire function. As usual, the Fatou and Julia sets of f

are denoted by $F(f)$ and $J(f)$; the *set of escaping points* of f is

$$I(f) := \{z \in \mathbb{C} \colon |f^n(z)| \to \infty\}.$$

The set of *singularities of* f^{-1}, denoted $\operatorname{sing}(f^{-1})$, consists of all finite critical and asymptotic values of f; the elements of $S(f) := \overline{\operatorname{sing}(f^{-1})}$ are called the *singular values* of f. We will mostly be interested in the aforementioned *Eremenko-Lyubich class*

$$\mathcal{B} := \{f \colon \mathbb{C} \to \mathbb{C} \text{ transcendental entire} \colon S(f) \text{ is bounded}\}.$$

An *(invariant) Siegel disk* of f is a simply connected component of $F(f)$ with $f(U) = U$ such that $f|_U$ is conjugate to an irrational rotation. A *periodic Siegel disk* is a component of $F(f)$ which is an invariant Siegel disk for some iterate of f.

We conclude any proof by the symbol $\blacksquare$. Proofs of separate claims within a larger proof will be completed by $\triangle$, while statements cited without proof are indicated by $\square$.

2. Tracts and Expansion

Throughout this section, we fix some entire function f in the Eremenko-Lyubich class $\mathcal{B}$. We will review some of the standard constructions used when dealing with such maps and deduce an expansion property (Lemma 2.3 below) which is essential for our arguments. We also recall some recent results on the existence of unbounded connected subsets of $I(f)$.

Tracts and Fundamental Domains. Define $K := 1 + \max\big(|f(0)|, \max_{s\in S(f)} |s|\big)$ and $G_K := \{z \in \mathbb{C} \colon |z| > K\}$. Then each component of $f^{-1}(G_K)$ is a simply connected domain whose boundary is a Jordan arc tending to infinity in both directions. These components are called the *tracts* of f; restricted to any tract, f is a universal covering onto G_K.

Let γ be a curve in G_K which does not intersect any tracts and which connects ∂G_K to ∞. (For example, we can let γ be a piece of the boundary of one of the tracts, together with a curve connecting it to ∂G_K if necessary.) Then $f^{-1}(\gamma)$ cuts every tract into countably many components, which we call *fundamental domains*; each fundamental domain maps univalently to $G_K \setminus \gamma$ under f. We would like to note the elementary but important fact that any bounded subset of $\mathbb{C}$ intersects at most finitely many fundamental domains of f.

Our goal in this section is to prove the following lemma. It basically states that a point whose orbit stays within finitely many fundamental domains escapes to infinity, provided it is large enough.

2.1. Lemma (Growth of orbits)**.**
Suppose $F_1, \dots, F_k$ are fundamental domains of f, and $R > 0$. Denote by X the set of all $z \in \mathbb{C}$ with the following property:

$$\textit{If } n \geq 0 \textit{ with } |f^m(z)| \geq R \textit{ for } m = 0, \dots, n, \textit{ then } f^n(z) \in \bigcup_{j=1}^{k} F_j.$$

Then there exists $R' > 0$ such that

$$X \cap \{|z| \geq R'\} \subset \{z \in I(f) \colon |f^n(z)| \geq R \textit{ for all } n \geq 0\}.$$

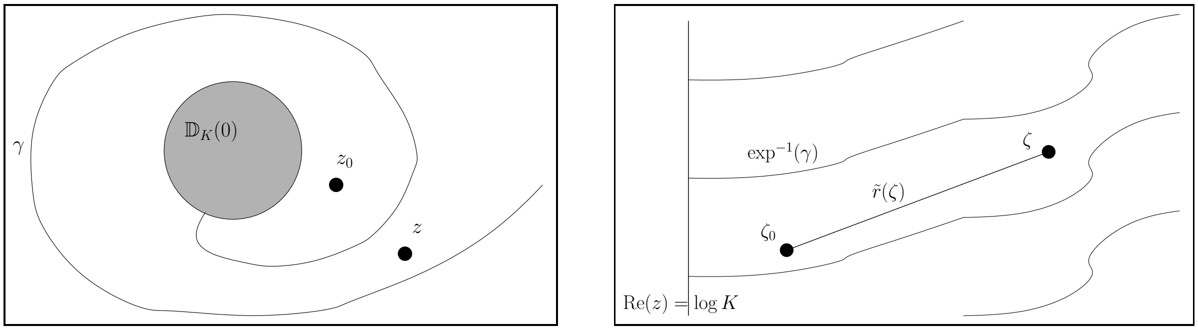


FIGURE 2. Definition of $r(z) = \widetilde{r}(\zeta_0)$. On the left, the picture is in the z-plane, while the right hand side is drawn in logarithmic coordinates.

Remark. X consists of all points which do not leave the union of $F_1, \dots, F_k$ without passing through the disk $\mathbb{D}_R(0)$ first; in particular, X contains this disk $\mathbb{D}_R(0)$ itself. X will generally not be invariant under f, but if $z \in X$ and $|z| \geq R$, then $f(z) \in X$.

We wish to note that the subsequent constructions and results, which lead to the proof of this lemma and are perhaps of a somewhat technical nature, are not required for the other sections of the article.

Logarithmic Coordinates. In [EL], functions in class $\mathcal{B}$ were studied by applying a logarithmic change of variable both on G_K and on the tracts of f. More precisely, let $H := \{\mathrm{Re}\, z > \log K\} = \exp^{-1}(G_K)$. Since $0 \notin f^{-1}(G_K)$, and since f is a universal covering on every tract, we can find a map Φ from the set $\mathcal{T} := \exp^{-1}(f^{-1}(G_K))$ to H with $\exp \circ \Phi = f \circ \exp$. If T is a component of $\mathcal{T}$, then T is simply connected and $\exp(T)$ is a tract of f; we call T a tract of Φ. The map $\Phi\colon T \to H$ is a conformal isomorphism for every tract T of Φ.

Note that $\exp^{-1}(\gamma)$ consists of countably many curves, which cut the half plane H into countably many *fundamental strips*. By definition, the boundaries of these strips do not intersect any tracts of Φ.

The following lemma — proved by a simple application of Koebe's $\frac{1}{4}$-theorem — provides a basic expansion estimate for Eremenko-Lyubich functions.

2.2. Lemma ([EL, Lemma 1])**.**
For any $\zeta \in \mathcal{T}$,

$$|\Phi'(\zeta)| \geq \frac{1}{4\pi}(\mathrm{Re}\, \Phi(\zeta) - \log K). \qquad \square$$

Growth of points in a fundamental domain. In order to prove Lemma 2.1, we wish to show that, for any fixed fundamental domain, every "sufficiently large" point has an even larger image. While we cannot expect this statement to hold when size is measured by the modulus of a point, we will associate a size $r(z)$ to a point z which makes it true.

In order to do this, let us fix any base point $z_0 \in G_{e^{16\pi}K} \setminus \gamma$ for the remainder of this section. If $z \in G_K \setminus \gamma$ and $\zeta \in \exp^{-1}(z)$, we define $r(z) := \widetilde{r}(\zeta) := |\zeta - \zeta_0|$, where ζ_0 is the unique point of $\exp^{-1}(z_0)$ which belongs to the same fundamental strip as ζ. (Compare Figure 2.) Note that $r(z) \geq \log|z| - \log|z_0|$, and that $r(z)$ remains bounded

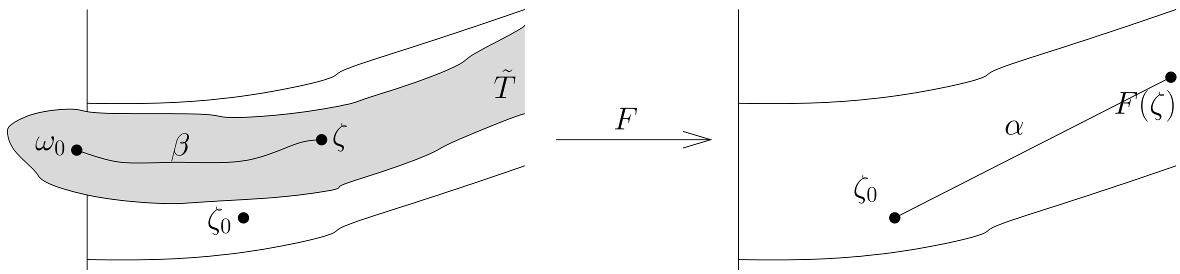


FIGURE 3. Proof of Lemma 2.3.

when z ranges over a bounded subset of $G_K \setminus \gamma$. (However, $r(z)$ may tend to infinity much faster than $\log|z|$).

2.3. Lemma (Expansion in fundamental domains)**.**
Let F be any fundamental domain. Then there exists some $C_F > 0$ with the following property. If $z \in F \cap G_K$ with $r(z) \geq C_F$, then $r(f(z)) \geq 2r(z)$.

Proof. The proof will take place completely in logarithmic coordinates. Let S be any fundamental strip and let ζ_0 be the unique point of $\exp^{-1}(z_0) \cap S$. Let $\widetilde{F}$ be the unique component of $\exp^{-1}(F)$ for which $S \cap \widetilde{F}$ is unbounded, and let $\widetilde{T}$ be the tract of Φ containing $\widetilde{F}$. Then Φ maps $\widetilde{F}$ conformally onto some fundamental strip; by postcomposing Φ with a suitable translation we may suppose, for simplicity, that $\Phi(\widetilde{F}) = S$. For the remainder of the proof, we denote the inverse $(\Phi|_{\widetilde{T}})^{-1}$ simply by Φ^{-1}.

Set $\omega_0 := \Phi^{-1}(\zeta_0) \in \widetilde{T}$ and define

$$C_1 := 1 + \sup\bigl\{\widetilde{r}(\zeta) : \zeta \in H \cap \widetilde{F} \text{ and } \operatorname{Re}\Phi(\zeta) < \log K + 16\pi\bigr\},$$
$$C_2 := 1 + \sup\bigl\{\widetilde{r}(\zeta) : \zeta \in (H \cap \widetilde{F}) \setminus S\bigr\}$$

and $C_F := \max(C_1, C_2, 2 \cdot |\omega_0 - \zeta_0|)$. We must show that

$$|\Phi(\zeta) - \zeta_0| \geq 2\widetilde{r}(\zeta)$$

for every point $\zeta \in \widetilde{F} \cap H$ with $\widetilde{r}(\zeta) \geq C_F$.

So suppose that ζ is such a point. Then, by definition of C_1 and C_2, $\zeta \in S$ and $\operatorname{Re}\Phi(\zeta) \geq \log K + 16\pi$. Let α denote the straight line segment connecting $\Phi(\zeta)$ and ζ_0, and set $\beta := \Phi^{-1}(\alpha)$. (See Figure 3.) Then, by Lemma 2.2, $|\Phi'(z)| \geq 4$ for every point $z \in \beta$, and thus

$$\ell(\beta) \leq \frac{1}{4}\ell(\alpha) = \frac{|\Phi(\zeta) - \zeta_0|}{4}.$$

Since β is a curve connecting ζ and ω_0, it follows that

$$\widetilde{r}(\zeta) = |\zeta - \zeta_0| \leq |\zeta - \omega_0| + |\omega_0 - \zeta_0| \leq \frac{|\Phi(\zeta) - \zeta_0|}{4} + \frac{\widetilde{r}(\zeta)}{2},$$

which means that $|\Phi(\zeta) - \zeta_0| \geq 2\widetilde{r}(\zeta)$, as required. ■

Proof of Lemma 2.1. By Lemma 2.3, for each j there exists some $C_j > 0$ such that $r(f(z)) \geq 2r(z)$ for every $z \in F_j \cap G_K$ with $r(z) \geq C_j$. Choose a $T > 0$ such that $|z| \geq R$ whenever $r(z) \geq T$ and set $C := \max\bigl(T, \max_j C_j\bigr)$.

If $z \in X$ with $r(z) \geq C$, then $f(z) \in X$ and $r(f(z)) \geq 2r(z)$. Indeed, we have $r(z) \geq T$, and hence $|z| \geq R$. By definition of X, this implies that $f(z) \in X$, and that $z \in F_j$ for some j. By the definition of C, we have $r(f(z)) \geq 2r(z)$ as claimed.

It follows inductively that $|f^n(z)| \geq R$ and $r(f^n(z)) \geq 2^n r(z)$ for all n; in particular, $z \in I(f)$. The claim follows by choosing $R' \geq K$ sufficiently large so that $r(z) \geq C$ whenever $|z| \geq R'$. ■

Continua Consisting of Escaping Points. It is an open question, posed by Eremenko [E], whether, for every transcendental entire function, every component of $I(f)$ is unbounded. Recently, Rippon and Stallard [RS] showed that every component of the set $A(f) \subset I(f)$ of "fast" escaping points, introduced by Bergweiler and Hinkkanen [BeH], is unbounded. For functions in class $\mathcal{B}$, their ideas can be used to obtain the more precise statement of the following theorem. This theorem will be used only in Section 4.

2.4. Theorem (Existence of unbounded connected sets)**.**
Let $f \in \mathcal{B}$, let F be a fundamental domain of f, and let $R > 0$. Then there exists an unbounded closed connected set L such that, for all $j \geq 0$,

$$f^j(L) \subset F \cap I(f) \cap \{z \colon |z| \geq R\}.$$

Remark. Since the preparation of this article, there have been several improvements on this result. For example, it follows from [R4] that the set L can be chosen to be forward invariant. Also, in [BRS] it is shown that any tract of any entire transcendental function (not necessarily in class $\mathcal{B}$) contains an unbounded closed connected set of points which escape within this tract. For the reader's convenience, we will nonetheless include the simple direct proof of the above theorem here.

Proof. Let C_F be the constant from Lemma 2.3, and let K again be as defined at the beginning of the section. We can choose $C > C_F$ large enough such that $|z| > \max(K+1, R)$ whenever $r(z) \geq C$, and such that F contains some point w_0 with $r(w_0) < C$. We define a sequence (U_j) by letting U_0 be the unbounded connected component of $\{z \in F : r(z) > C\}$, and denoting the unbounded connected component of $f(U_j) \cap F$ by U_{j+1} for each $j \geq 0$. (Note that such a component exists and is unique: in fact, by induction the set U_j contains all points of F of sufficiently large modulus.)

Since $U_{j+1} \subset f(U_j)$ (and $f|_{U_j}$ is univalent), we can also define

$$V_j := (f|_{U_0})^{-1}((f|_{U_1})^{-1}(\dots(f|_{U_{j-1}})^{-1}(U_j)\dots)) = (f|_F)^{-n}(U_j),$$

for all $j \geq 0$. Then V_j is connected and $f^j : V_j \to U_j$ is a conformal isomorphism.

By choice of C, every point $z \in U_j$ satisfies $r(z) > 2^j C$ and $|z| > \max(K+1, R)$. Thus

$$r(f^k(z)) > 2^k \cdot C \quad \text{and} \quad |f^k(z)| > R \tag{1}$$

for every $z \in V_j$ and $k = 0, \dots, j$.

Claim. For every $j \geq 0$, there exists $z \in \partial V_j$ with $r(z) = C$.

Proof. By choice of C, the point $w_0 \in F$ does not belong to U_j, and thus there is some point $z_0 \in F \cap \partial U_j$. Since $|z_0| \geq K+1$, this means that $z_0 \in f(F) \cap \partial f(U_{j-1})$. In other words, $(f|_F)^{-1}(z_0) \in F \cap \partial U_{j-1}$.

Continuing inductively, the (unique) point $z \in \partial V_j$ with $f^j(z) = z_0$ satisfies $z \in F \cap \partial U_0$, and therefore $r(z) = C$. $\triangle$

Since the set $\{z \in \mathbb{C} : r(z) = C\}$ is bounded, $\widetilde{L} := \bigcap_{j\geq 0} \overline{V_j}$ is nonempty (as well as invariant, closed and unbounded). By (1), $\widetilde{L}$ satisfies all requirements of the theorem, except that it need not be connected. However, $\widetilde{L} \cup \{\infty\}$ is compact and connected, and we can complete the proof by letting L be any connected component of $\widetilde{L}$. ■

3. Proof of the main theorem

3.1. Definition (Extendable sets)**.**
Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. $A \subset \mathbb{C}$ is called an extendable set *(for f) if, whenever z tends to ∞ in A, $|f(z)|$ also tends to ∞. (In particular, every bounded set is extendable.)*

Our main result below will apply in any case where a Siegel disk, fixed ray etc. is an extendable set. In order to obtain the theorems as stated in the introduction, we remark the following.

3.2. Observation (Sufficient conditions for extendability)**.**
Let $f\colon \mathbb{C} \to \mathbb{C}$ be entire and let $A \subset \mathbb{C}$.

(a) *A is extendable if and only if $\overline{A}$ is.*
(b) *If $f|_A$ is injective, then A is extendable if and only if ∂A is.*
(c) *If A is closed and $f\colon A \to \overline{f(A)}$ is a proper map (in particular, if $f\colon A \to \overline{f(A)}$ is a homeomorphism), then A is extendable.*

Proof. (a) is trivial, and (c) follows from the fact that every proper map between two topological spaces extends continuously to a map between their one-point compactifications.

To prove (b), suppose that ∂A is extendable and let (z_n) be a sequence in A such that $f(z_n) \to w_0 \in \mathbb{C}$. We need to show that z_n is bounded. By hypothesis, there exists some $R > 0$ such that $|f(z)| > |w_0| + 1$ for all $z \in \partial A$ with $|z| \geq R$. Let J be some Jordan curve in $\mathbb{C} \setminus f^{-1}(w_0)$ which surrounds $\mathbb{D}_R(0)$. If all but finitely many z_n are surrounded by J, there is nothing to prove.

Otherwise, set $\varepsilon := \min\big(1, \operatorname{dist}(w_0, f(J))\big)$ and let z_k be a point on the outside of J with $f(z_k) \in \mathbb{D}_\varepsilon(w_0)$. Then the component U of $f^{-1}(\mathbb{D}_\varepsilon(w_0))$ containing z_k does not intersect J, and is thus contained in $\mathbb{C} \setminus \mathbb{D}_R(0)$. By choice of R, it follows that $U \cap \partial A = \emptyset$, and thus $U \subset A$. Since $f|_U$ is injective, it follows easily that $f\colon U \to \mathbb{D}_\varepsilon(w_0)$ is a conformal map, and thus $z_n \to (f|_U)^{-1}(w_0)$ by injectivity of $f|_A$. This means that (z_n) is bounded, as required. ■

Remark 1. If A is closed and connected, A contains no critical points and $f\colon A \to \overline{f(A)}$ is a homeomorphism, then it is easy to see that there exists a $\mathbb{C}$-neighborhood U of A such that $f|_U$ is univalent. For this reason, we call a set A satisfying the above assumptions a *set of univalence*. If $f \in \mathcal{B}$ and A is a set of univalence for f, then one can show that the branch $\varphi := (f|_A)^{-1}$ can actually be defined on a large domain of a particularly nice

form, namely one whose complement is a union of finitely many arcs to ∞ and finitely many compact connected sets. However, we do not require this fact in this article.

Remark 2. Suppose again that A is closed, connected and contains no critical points. If f is a polynomial, or if A is bounded, then A is a set of univalence if and only $f|_A$ is injective. This is, of course, no longer true for entire functions: consider e.g. $f := \exp$ and $A := \mathbb{R}$. For an example where $f\colon A \to \overline{f(A)}$ is bijective but not a homeomorphism, consider the map $f(z) := z\exp(z)$. Here 0 is both a parabolic fixed point and an asymptotic value. Let B be a Jordan curve through 0 which surrounds the critical value $-1/e$. If we let A be the curve obtained by analytic continuation of the branch φ of f^{-1} with $\varphi(0) = 0$ along B, then A is a Jordan arc from 0 to ∞ and $f\colon A \to B$ is bijective. If B was chosen to be e.g. the boundary of an attracting petal at 0, then we can set $A' := \bigcup_{j\geq 0} f^n(A)$. This set A' is closed and forward invariant, and $f\colon A' \to \overline{f(A')}$ is bijective (see Figure 4(a)).

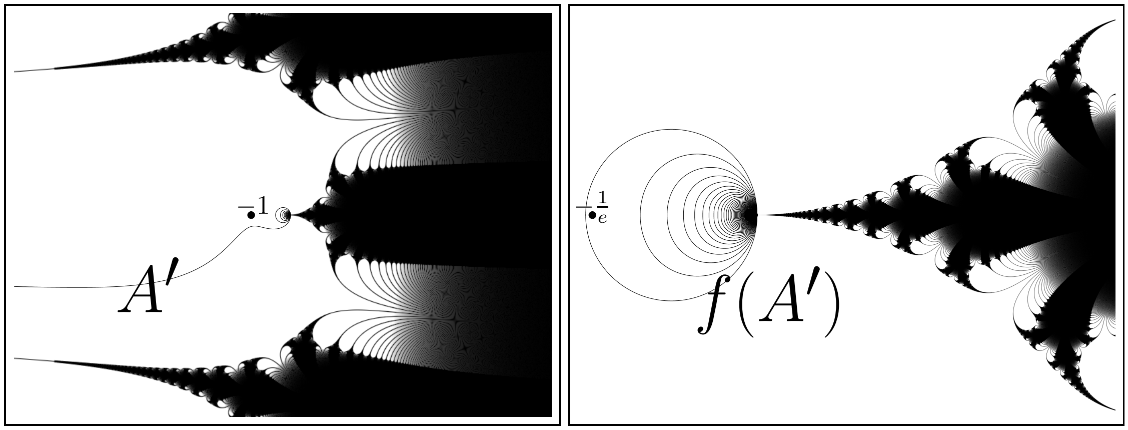
This last example is perhaps not quite satisfactory since the set A' contains a singular value. This can be avoided by a simple modification. Indeed, let $f(z) := 1/4\big((z+1)e^z - 1\big)$ (which is obtained from our previous example by affine coordinate changes in the domain and range). Here the critical point $c = -2$ and the asymptotic value $a = -1/4$ both belong to the immediate basin of 0 (which is the entire Fatou set). We can let B consist of a small circle around a, together with a curve spiralling in towards this circle in both directions, and surrounding the critical value $f(c)$ (see Figure 4(b)). Then, as above, there is an unbounded set A which is mapped bijectively to B, and if B was chosen correctly, then A and B are disjoint. Connecting A and B by an interval of the real axis, and adding all forward iterates as well as the fixed point 0, we obtain a closed invariant set A' which contains no singular values and for which $f\colon A' \to \overline{f(A')}$ is bijective; see Figure 4(b).

We are now ready to prove Theorem 5. In fact, we show the following more general result.

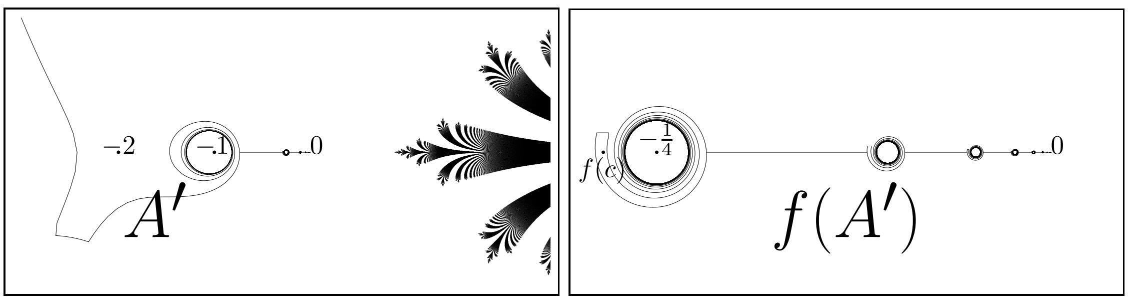
3.3. Theorem (Invariant extendable sets)**.**
Let $f \in \mathcal{B}$ and let $A \subset \mathbb{C}$ be an extendable set.

(a) *Suppose that A is connected and $f(A) \subset A$. Then for every $R > 0$, there exists some $R' > 0$ such that*
$$A \cap \{|z| \geq R'\} \subset \{z \in I(f)\colon |f^n(z)| \geq R \text{ for all } n \geq 0\}.$$

(b) *More generally, suppose that $C \subset A \cap f^{-1}(A)$ is connected. Let $R > 0$, and let X denote the set of all $z \in C$ with the following property:*
$$\text{If } n \geq 0 \text{ with } |f^m(z)| \geq R \text{ for } m = 0, \dots, n, \text{ then } f^n(z) \in C.$$
Then there exists $R' > 0$ such that
$$X \cap \{|z| \geq R'\} \subset \{z \in I(f)\colon |f^n(z)| \geq R \text{ for all } n \geq 0\}.$$



(a) $f(z) = z\exp(z)$



(b) $f(z) = 1/4((z+1)\exp(z) - 1)$

FIGURE 4. Examples of closed connected invariant sets A' which are mapped bijectively but not homeomorphically by an entire function f, as described in Remark 2 after Observation 3.2. In both cases, the compact image $f(A') \subset A'$ is shown in greater magnification on the right.

Proof. (a) follows from (b) by setting $C := A$. Thus, it suffices to prove (b). We claim that there is $T > 0$ such that $\{z \in C\colon |z| \geq T\}$ is contained in finitely many fundamental domains of f.

To prove this, set $K := 1 + \max\big(|f(0)|, \max_{s\in S(f)} |s|\big)$ as before, and let γ be the curve to ∞ used in the definition of fundamental domains. Recall that $f(\gamma) \subset \overline{\mathbb{D}_K(0)}$. A is extendable, so we can find $T \geq T_1 > 0$ such that, for all $z \in A$,

$$\begin{aligned} |z| \geq T_1 &\implies |f(z)| > K \quad \text{and} \\ |z| \geq T &\implies |f(z)| \geq T_1. \end{aligned}$$

If $z \in A \cap f^{-1}(A)$ with $|z| \geq T$, then $|f(z)| > K$, and thus z belongs to some tract of f. However, also $|f(f(z))| > K$, and in particular $f(z) \notin \gamma$; thus z belongs to some fundamental domain of f. In other words, every component of $C \setminus D_T(0)$ is contained in a single fundamental domain. We may suppose that T was chosen large enough such that $C \cap \mathbb{D}_T(0) \neq \emptyset$. Since C is connected, it follows that every fundamental domain which meets $C \setminus \mathbb{D}_T(0)$ intersects the circle $\{|z| = T\}$. As remarked in Section 2, there are only finitely many fundamental domains with this property.

Let $F_1, \dots, F_k$ denote these fundamental domains. Let $z \in X$ and let $n \geq 0$ with $|f^m(z)| \geq \max(T, R)$ for all $m \in \{0, \dots, n\}$. Then, by definition of X,

$$f^n(z) \in C \cap \{|z| \geq T\} \subset \bigcup F_j.$$

By Lemma 2.1, there exists $R' > 0$ such that

$$X \cap \{|z| \geq R'\} \subset \{z \in I(f) \colon |f^n(z)| \geq \max(T, R) \text{ for all } n\}. \qquad \blacksquare$$

Since Siegel disks never contain escaping points, we have the following immediate corollary.

3.4. Corollary (Extendable Siegel disks are bounded)**.**
Let $f \in \mathcal{B}$, let U be a Siegel disk of f and suppose that U is an extendable set. Then U is bounded. $\blacksquare$

Proof of Theorem 1. If $f : \partial U \to \partial U$ is a homeomorphism, this implies that $\overline{U}$ (and hence U) is an extendable set by Observation 3.2. Theorem 1 now follows from Corollary 3.4. $\blacksquare$

3.5. Corollary (Bounded accumulation sets)**.**
Let $f \in \mathcal{B}$ and let $\gamma \colon (-\infty, 1] \to \mathbb{C}$ be a curve with $f(\gamma(t)) = \gamma(t+1)$ for all $t \leq 0$. If γ is an extendable set, then γ is bounded.

Proof. We will apply Theorem 3.3 (b) to the sets $A := \gamma\big((-\infty, 1]\big)$ and $C := \gamma\big((-\infty, 0]\big)$. If $z \in C$ such that $f(z) \notin C$, then $f(z) \in \gamma\big([0,1]\big)$. Thus, if we set $R := 1 + \max_{t \in [0,1]} |\gamma(t)|$, then the set X as defined in Theorem 3.3 (b) is all of C. Since every point of C eventually maps to $\gamma\big([0,1]\big)$ under iteration, this means that there is $R' > 0$ such that $C \subset \overline{\mathbb{D}_{R'}(0)}$. $\blacksquare$

4. Obtaining Extendability

It is often difficult to show that a given set is extendable. In this section, we describe how to do this in the case where all singular values (with the possible exception of one critical value) are contained in the Julia set. The main idea is to apply Theorem 2.4 and the following observation.

4.1. Lemma (Obtaining extendability)**.**
Let $f \in \mathcal{B}$. Let $A \subset \mathbb{C}$ be a connected set and set $B := \overline{f(A)}$.

(a) *Suppose that there exists some connected open set $G \subset A$ such that $f|_G$ is univalent and $B \cap S(f) \subset f(G)$. Suppose furthermore that ∞ is accessible from every component W of $\mathbb{C} \setminus B$ for which $W \cap S(f) \neq \emptyset$. Then $f \colon \overline{A} \to B$ is a homeomorphism.*

(b) *Suppose that there exists some isolated point $s_0 \in S(f)$ which is a critical (but not an asymptotic) value such that $B \cap (S(f) \setminus \{s_0\}) = \emptyset$ and such that ∞ is accessible from every component W of $\mathbb{C} \setminus B$ with $W \cap (S(f) \setminus \{s_0\}) \neq \emptyset$. Then A is an extendable set.*

Proof. Set $S_0 := f(G)$ for (a) and $S_0 := \{s_0\}$ for (b). Let $\mathcal{W}$ denote the set of those components W of $\mathbb{C} \setminus B$ which satisfy $W \cap (S(f) \setminus S_0) \neq \emptyset$. Note that $\mathcal{W}$ is an open cover of the compact set $S(f) \setminus S_0$ and contains no proper subcover. Hence $\mathcal{W}$ is finite.

Now, for each component $W \in \mathcal{W}$, we can choose some compact, connected and full (i.e., non-separating) subset $K_W \subset W$ with $S(f) \cap W \subset K_W$, and a curve $\gamma_W \subset W \setminus K_W$ with one endpoint in K_W and the other at ∞. Then the set $V := \mathbb{C} \setminus \bigcup_{W \in \mathcal{W}} (K_W \cup \gamma_W)$ is simply connected, contains B and is disjoint from $S(f) \setminus S_0$. Let U be the component of $f^{-1}(V)$ which contains A.

In case of (a), we claim that $f\colon U \to V$ is a conformal isomorphism. Indeed, if $G = \emptyset$ (which is the only case in which we will apply this lemma), then this follows by the monodromy theorem. Otherwise, it is not difficult to see that the branch $(f|_G)^{-1}$ extends to a branch of f^{-1} on V, whose image is necessarily U.

In case of (b), $f\colon U \to V$ is either conformal or a finite-degree covering with a single branched point. In either case, $f\colon \overline{A} \to B$ is a proper map, which implies that $\overline{A}$ (and thus A) is extendable. ■

4.2. Theorem (Accessibility of infinity)**.**
Let $f \in \mathcal{B}$ and let $U \subset \mathbb{C}$ be connected. Suppose that there are three fundamental domains F_1, F_2, F_3 of f such that

$$U \cap \{z \in I(f)\colon f^k(z) \in F_i \text{ for all sufficiently large } k\} = \emptyset$$

for all $i \in \{1, 2, 3\}$. Then ∞ is accessible from every component W of $\mathbb{C} \setminus \overline{U}$ which satisfies $W \cap J(f) \neq \emptyset$.

Proof. By Theorem 2.4, for each $i \in \{1, 2, 3\}$, there is an unbounded closed connected set $C_i \subset I(f)$ with $f^k(C_i) \subset F_i$ for all $k \geq 0$.

Let $z_0 \in W \cap J(f)$, and set $\delta := \operatorname{dist}(z_0, \overline{U})$ and $D := \mathbb{D}_{\frac{\delta}{2}}(z_0)$. Let $i \in \{1, 2, 3\}$. Then, since $z_0 \in J(f)$, there exists some large n such that $f^n(D) \cap C_i \neq \emptyset$. Let A_i' be a component of $f^{-n}(C_i)$ with $A_i' \cap D \neq \emptyset$.

Note that the three sets A_i' are unbounded and pairwise disjoint. For each i, let A_i be the closure of some unbounded component of $A_i' \setminus D$. (Such a component exists by a simple application of the boundary bumping theorem [N, Theorem 5.6].) These components are disjoint from U by the assumption on U.

It follows that there exists $j \in \{1, 2, 3\}$ with $A_j \subset W$ (see Lemma A.2). By Lemma A.1, ∞ is accessible from W. ■

Proof of Theorem 2. If U is a Siegel disk, then U does not intersect $I(f)$ and therefore satisfies the assumption of Theorem 4.2. Thus ∞ is accessible from every component of $\mathbb{C} \setminus \overline{U}$ which intersects the Julia set. If $S(f) \subset J(f)$ and $S(f) \cap \partial U = \emptyset$, then we can apply Lemma 4.1 (a) to see that $f\colon \overline{U} \to \overline{U}$ is a homeomorphism. ■

Proof of Theorem 3. Let $U_0 \mapsto U_1 \mapsto \ldots \mapsto U_m \mapsto U_0$ be a cycle of Siegel disks for a map f with two critical values and no asymptotic values. Suppose that, for each j, at least one of the two critical values of f belongs to $J(f) \setminus \partial U_j$. (This assumption is automatically satisfied if no ∂U_j contains a critical value. Indeed, each Siegel disk boundary is contained in the postcritical set and f has no wandering domains [EL], so at least one critical value must belong to the Julia set.)

Applying Theorem 4.2 and Lemma 4.1 (b), we see that U_j is an extendable set for each j. Thus U_0 is an extendable set for f^m, and the claim follows from Corollary 3.4. ∎

For cases in which it may not be possible to control the eventual behavior of points, let us also show the following variant of Theorem 4.2.

4.3. Theorem (Accessibility when $S(f) \subset J(f)$)**.**
Let $f \in \mathcal{B}$ and suppose that $S(f) \subset J(f)$. Then for every $R > 0$ and $\varepsilon > 0$, there exists an n_0 with the following property. If

$$U \subset \{z \in \mathbb{C} \colon \exists n \geq n_0 \colon |f^n(z)| < R\}$$

is connected with $\operatorname{dist}(U, S(f)) \geq \varepsilon$*, then ∞ is accessible from every component of $\mathbb{C} \setminus \overline{U}$.*

Proof. Applying Theorem 2.4 to three different fundamental domains of f, there exist three unbounded closed connected sets

$$C_1, C_2, C_3 \subset \{z \in I(f) \colon |f^m(z)| \geq R \text{ for all } m\}$$

such that $f^m(C_i) \cap f^n(C_j) = \emptyset$ whenever $i \neq j$ and $m, n \geq 0$. Let

$$\mathcal{K} := \bigcup_{s \in S(f)} \overline{\mathbb{D}_{\frac{\varepsilon}{2}}(s)}.$$

As in the proof of Lemma 4.1, this set has finitely many components. For each such component K and each $i \in \{1, 2, 3\}$, we can again find some $n_{K,i}$ and a closed unbounded set $A_{K,i}$ connecting K to ∞ such that $f^{n_{K,i}}(A_{K,i}) \subset C_i$.

Let $n_0 := \max\limits_{K,i} n_{K,i}$. Then every set U as in the statement of the theorem is disjoint from $\mathcal{K} \cup \bigcup_{K,i} A_{K,i}$, and the proof proceeds as in Theorem 4.2. ∎

4.4. Corollary (Nonsingular rays are bounded)**.**
Let $f \in \mathcal{B}$ with $S(f) \subset J(f)$. Suppose that $\gamma \colon (-\infty, 1] \to \mathbb{C}$ is a curve with $f(\gamma(t)) = \gamma(t+1)$ for all $t \leq 0$ and $\overline{\gamma} \cap S(f) = \emptyset$. Then there exists $t \leq 0$ such that

$$f \colon \overline{\gamma\big((-\infty, t]\big)} \to \overline{\gamma\big((-\infty, t+1]\big)}$$

is a homeomorphism. In particular, γ is bounded.

Proof. Let $R := 1 + \max_{t \in [0,1]} |\gamma(t)|$ and $\varepsilon := \operatorname{dist}(\gamma, S(f))$. By Theorem 4.3, there exists $t := -n_0$ such that $\gamma\big((-\infty, t]\big)$ satisfies the assumptions of Lemma 4.1 (a), proving that f is a homeomorphism on its closure. That γ is bounded now follows from Corollary 3.5 and Observation 3.2. ∎

5. Landing of Periodic Rays

The classical snail lemma [M, Lemma 16.2] states that the landing point of an invariant curve for a holomorphic mapping f cannot be an irrationally indifferent fixed point. We will prove a generalization of this fact which allows us to prove the landing of certain periodic rays. Our methods are quite similar to those used by Perez-Marco in his study of hedgehogs [Pé]; since first preparing this article we have also learned that Risler [Ris] also studied completely invariant compact sets of univalent functions using similar considerations.

Let us say that a pair (f, K) of a compact set K and a holomorphic map f defined in a neighborhood U of K *has the snail lemma property* if every curve $\gamma\colon (-\infty, 1] \to U \setminus K$ with $f(\gamma(t)) = \gamma(t+1)$ and $\lim_{t\to-\infty} \operatorname{dist}(\gamma(t), K) = 0$ lands at a repelling or parabolic fixed point of f.

5.1. Lemma (Univalent snail lemma for the circle)**.**
Let U be a neighborhood of S^1, and let $f\colon U \to \mathbb{C}$ be a univalent function with $f(S^1) = S^1$. Then (f, S^1) has the snail lemma property.

Proof. Let γ be a curve as in the snail lemma property. Then f is not of finite order, as otherwise no such curve γ can exist. We may also assume (by reflection in S^1) that $\gamma \subset \mathbb{D}$ and (by restriction of U) that f has no fixed points outside the unit circle. Let us consider two cases.

First case: f possesses at least one fixed point. Since $f \neq \mathrm{id}$, the number of fixed points of f is finite. The unit circle S^1 is invariant under f, and thus these fixed points must be either attracting, repelling or parabolic. They cut the circle into finitely many intervals, and points in such an interval converge to one endpoint under forward iteration and to the other under backwards iteration. Thus, every interval, with the exception of one endpoint, is contained in the basin of attraction (or repulsion) of the other endpoint.

The accumulation set of γ cannot intersect any of the basins of attraction, because every point on γ eventually maps to $\gamma\big([0,1]\big)$. Since the accumulation set is connected, it consists of a single repelling or parabolic point, as required.

Second case: f has no fixed points. The argument in this case is completely analogous to the proof of the classical snail lemma. We shall therefore omit some of the details in the proof.

We may assume that $0 \notin U$. Set $\widetilde{U} := \exp^{-1}(U \cap \mathbb{D})$ and let $\widetilde{\gamma} \subset \widetilde{U}$ be any lift of γ under exp. We can then choose a lift $\widetilde{f}$ of f such that $\widetilde{f}(\widetilde{\gamma}(t)) = \widetilde{\gamma}(t+1)$.

Since f, and thus $\widetilde{f}$, has no fixed points,

$$c := \max_{r\in\mathbb{R}} |\operatorname{Im} \widetilde{f}(ir) - r| > 0.$$

Choose $\varepsilon > 0$ such that $|\operatorname{Im} \widetilde{f}(z) - \operatorname{Im} z| \geq \frac{c}{2}$ whenever $|\operatorname{Re} z| \leq \varepsilon$. To fix ideas, let us assume that $\operatorname{Im} \widetilde{f}(ir) < r$ for all $r \in \mathbb{R}$. It follows that $\vartheta(t) \to +\infty$ as $t \to -\infty$, where $\vartheta(t) := \operatorname{Im} \widetilde{\gamma}(t)$.

We can thus pick some $t_0 < 0$ such that $|\operatorname{Re} \widetilde{\gamma}(t)| \leq \varepsilon$ for $t \leq t_0$ and $\vartheta(t) \leq \vartheta(t_0)$ for $t \geq t_0$. Let $\widetilde{V}$ be the component of

$$\{z \in \widetilde{U} \setminus \widetilde{\gamma} : \operatorname{Im} z > \vartheta(t_0)\}$$

whose boundary contains the line $\{ir\colon r \geq \vartheta(t_0)\}$. It follows easily that $\widetilde{V} \subset \widetilde{f}(\widetilde{V})$.

Therefore $V := \exp(\widetilde{V})$ is a one-sided neighborhood of S^1 with $f^{-1}(V) \subset V$. Since the boundary of V is contained in U, the iterates $f^{-n}|_V$ converge locally uniformly to a fixed point of f by [M, Lemma 5.5]. This contradicts our assumption. ■

5.2. Lemma (General univalent snail lemma)**.**
Let $K \subset \mathbb{C}$ be compact and connected, and suppose that f is a function univalent in a neighborhood U of K, with $f(K) = K$. Then (f, K) has the snail lemma property.

Proof. Let $\gamma \subset U \setminus K$ be a curve as in the definition of the snail lemma property and let V be the component of $\mathbb{C} \setminus K$ containing γ. Then $U \cap V$ is connected and is thus mapped by f into some component of $\mathbb{C} \setminus K$. Since $f(U \cap V)$ contains γ and thus intersects V, it follows that $f(U \cap V) \subset V$.

Let $\varphi\colon V \to \mathbb{D}$ be a Riemann mapping of V, and define

$$g\colon \varphi(U \cap V) \to \mathbb{D}; z \mapsto \varphi(f(\varphi^{-1}(z))).$$

Since f is continuous in a neighborhood of K and $f^{-1}(K) = K$, every prime end of K is mapped to a prime end of K by f. Thus g extends continuously to S^1 by Carathéodory's Theorem [P2, Theorem 2.15]. By the Schwarz Reflection Principle [A], g extends to an analytic function on a neighborhood of S^1.

This extended function g is univalent, and thus the curve $\varphi(\gamma)$ lands at a repelling or parabolic fixed point z_0 of g by Lemma 5.1. Let D be a linearizing neighborhood or repelling petal of z_0 which is compactly contained in the domain of definition of g and which contains some end piece of $\varphi(\gamma)$.

Then $\varphi^{-1}(D)$ is invariant under f^{-1} and contains an end piece of γ. Again, the functions $(f|_D)^{-n}$ converge locally uniformly to a fixed point of f in K by [M, Lemma 5.5]. It follows that γ lands at this fixed point, which is repelling or parabolic by the classical Snail Lemma. ■

5.3. Corollary (Landing of univalent rays)**.**
Let f be an entire function and suppose that $\gamma\colon (-\infty, 1] \to I(f)$ is a curve with $f(\gamma(t)) = \gamma(t+1)$. Suppose furthermore that γ does not accumulate at any critical point of f and that, for some $t \leq 0$, the restriction

$$f\colon \overline{\gamma\big((-\infty, t]\big)} \to \overline{\gamma\big((-\infty, t+1]\big)}$$

is a homeomorphism. Then γ lands at a repelling or parabolic fixed point of f.

Proof. By Corollary 3.5, the accumulation set K of γ is bounded, and therefore contains no escaping points. In particular, $\gamma \cap K = \emptyset$. Since $f\colon K \to K$ is a homeomorphism and since K contains no critical points of f, it follows easily that there exists some neighborhood U of K such that $f|_U$ is univalent (see [H, Lemma 3]). The claim now follows from Lemma 5.2. ■

Proof of Theorem 4. If we are in the first case of Theorem 4, then Corollary 3.5 implies (using Observation 3.2) that the accumulation set of γ is bounded. In the second case, the same follows from Corollary 4.4. So in either case Corollary 5.3 implies that γ lands at a repelling or parabolic fixed point of f, as claimed. ■

In [R3], it was shown that periodic rays of exponential maps *always* land, but this requires the "lambda-lemma" and deep results on exponential parameter space. We can now deduce a special case of this theorem without requiring any parameter-space arguments, but also without any a priori assumptions on hyperbolic expansion. (Compare Appendix B).

5.4. Corollary (Nonsingular exponential rays)**.**
Suppose that $f(z) = \exp(z) + \kappa$ and that $\gamma\colon (-\infty, \infty) \to \mathbb{C}$ is a periodic dynamic ray with $\kappa \notin \bigcup_j \overline{f^j(\gamma)}$. Then γ lands at a repelling or parabolic periodic point of f.

Proof. First suppose that $\kappa \in F(f)$. Since f (like all maps with only finitely many singular values) has no wandering domains [EL], this implies that κ belongs to the basin of an attracting or parabolic periodic point. In this case, all periodic rays of f land by Corollary B.4.

So now suppose that $\kappa \in J(f)$. Let n be the period of γ. Let N be large enough and set

$$g_j := \overline{f^j\big(\gamma\big((-\infty, -N]\big)\big)}$$

By Theorem 4.3 and Lemma 4.1, if N was chosen large enough, then $f\colon g_j \to g_{j+1}$ is a homeomorphism for $j = 0, \dots, n$. Therefore $f^n\colon g_0 \to g_n$ is a homeomorphism, and the claim follows by applying Corollary 5.3 to f^n. ■

Appendix A. Two topological facts

This section is dedicated to proving the two simple topological facts which were used in Section 4.

Lemma A.1 (Accessibility criterion)**.**
Let U be a domain in $\mathbb{C}$. Suppose that there exists an unbounded closed connected set $A \subset U$. Then ∞ is accessible from U.

Proof. For each $z \in A$, let $\delta(z) := \min(\operatorname{dist}(z, \partial U), 1)$. For every $n \in \mathbb{N}$, the set

$$A_n := \{z \in A\colon n - 1 \leq |z| \leq n\}$$

is compact; thus there exists a finite set $K_n \subset A_n$ such that

$$A_n \subset \bigcup_{z \in K_n} \mathbb{D}_{\delta(z)}(z).$$

We claim that there is an infinite sequence $z_1, z_2, z_3, \dots \in K := \bigcup_j K_j$ with the property that $\mathbb{D}_{\delta(z_j)}(z_j) \cap \mathbb{D}_{\delta(z_{j+1})}(z_{j+1}) \neq \emptyset$ for all j and such that $z_j \neq z_{j'}$ for all $j \neq j'$. Indeed, consider the (infinite) graph on the set K in which z and w are adjacent if $\mathbb{D}_{\delta(z)}(z) \cap \mathbb{D}_{\delta(w)}(w) \neq \emptyset$. Then the graph G is connected and contains arbitrarily long paths. By König's lemma (see e.g. [D, Lemma 7.1.3]), G contains an infinite path, as desired.

So let $z_1, z_2, \dots$ be a sequence as above. Then the curve obtained by connecting every z_j to z_{j+1} by a straight line segment is a curve to ∞ in U. ■

Lemma A.2 (Separation lemma)**.**
Let $D \subset \mathbb{C}$ be a closed disk, and let $U \subset \mathbb{C}$ be connected with $\operatorname{dist}(U, D) > 0$.

Suppose that C_1, C_2, C_3 are pairwise disjoint closed connected sets such that $C_j \cap U = \emptyset$, $C_j \cap D \neq \emptyset$ and $C_j \setminus D$ is connected for every j. Then there exists $i \in \{1, 2, 3\}$ such that $C_i \cap \overline{U} = \emptyset$.

Proof. If, for some $i_1, i_2 \in \{1, 2, 3\}$, the sets U and $C_{i_1} \setminus D$ belong to different components of $\mathbb{C} \setminus (C_{i_2} \cup D)$, then $C_{i_1} \cap \overline{U} = \emptyset$, and we are done. Otherwise, let $K_i := \hat{\mathbb{C}} \setminus G_i$, where G_i is the component of $\mathbb{C} \setminus (C_i \cup D)$ containing U. Then each K_i is a nonseparating continuum, and any two of these intersect exactly in $D \cup \{\infty\}$.

The set $\hat{\mathbb{C}}\setminus\bigcup K_i$ has exactly three components W_1, W_2, W_3 [St], which we may suppose labelled such that $U \subset W_1$. It is a simple application of Janiczewski's theorem (compare [P1, Theorem 1.9]) that there exist $i, j \in \{1, 2, 3\}$, $i \neq j$, such that $K_i \cup K_j$ does not separate W_2 and W_3. By relabelling, we may assume that $i = 1$ and $j = 2$.

Let V be the component of $\hat{\mathbb{C}} \setminus (K_1 \cup K_2)$ which contains W_2 and W_3, and define $K_3' := K_3 \setminus (D \cup \{\infty\})$. By construction, K_3' is connected, intersects V and is disjoint from $K_1 \cup K_2$. Thus $K_3 \subset V$. Also note that $V \cap W_1 = \emptyset$, since $K_1 \cup K_2$ separates the Riemann sphere by [St]. Therefore

$$C_3 \subset K_3' \cup D \subset V \cup D \subset (\mathbb{C} \setminus \overline{W_1}) \cup D \subset \mathbb{C} \setminus \overline{U}. \qquad \blacksquare$$

Appendix B. Hyperbolic Contraction

For completeness, let us discuss here in which situations hyperbolic contraction can be used to show the landing of fixed rays of an entire function. The proof in the polynomial case (Corollary B.5) was given by Douady and Hubbard [DH]. The following is something of a "folk theorem"; special cases with essentially the same proof can be found e.g. in [F, R1, S2, SZ].

Theorem B.1 (Landing of fixed rays via contraction)**.**
Let X be a Riemann surface, and let $U \subset X$ be a hyperbolic domain. Let $h : U \to X$ be holomorphic such that $h(U) \supset U$ and $h : U \to h(U)$ is a covering map. Furthermore suppose that h is not an irrational rotation of a disk, punctured disk or annulus, and that $h^n \neq \mathrm{id}$ for $n \geq 1$.

Let $\gamma\colon (-\infty, 1] \to U$ be a curve with $h(\gamma(t)) = \gamma(t + 1)$ for all $t \geq 1$. Then every accumulation point of γ in

$$U \cup \big\{z \in \partial U\colon h \text{ extends continously to } U \cup \{z\}\big\}$$

is a fixed point of (the extension of) h.

Proof. Denote hyperbolic distance and length in U (compare [M, Chapter 2]) by dist_U and ℓ_U, respectively. Since h is a covering map, h is a local isometry between U and $h(U)$ (with their respective hyperbolic metrics). By the Schwarz Lemma, the inclusion $U \to h(U)$ does not expand the hyperbolic metric, and so h is either a local isometry (if $U = h(U)$) or otherwise strictly expands the hyperbolic metric of U.

Let us first suppose that $\gamma|_{[0,1]}$ is smooth; at the end of the proof we will sketch how to deal with the general case. For $t \leq 0$, let us denote by ℓ_t the hyperbolic length of $\gamma([t, t + 1])$ in U. Since pullbacks under h contract the hyperbolic metric, ℓ_t is a nonincreasing function of t.

Suppose that $z_0 \in \partial U$ is an accumulation point of γ; say $\gamma(t_n) \to z_0$ with $t_n \to -\infty$. Then the hyperbolic distance between $z_n := \gamma(t_n)$ and $h(z_n) = \gamma(t_n + 1)$ is at most $\ell_{t_n} \leq \ell_0$; since the hyperbolic metric of U blows up near ∂U, it follows that the distance between z_n and $h(z_n)$ in X tends to 0. If h can be extended continuously into z_0, then $|z_0 - h(z_0)| = 0$ by continuity.

Now suppose that $h(U) \supsetneq U$ and let z_0 be an accumulation point of γ in U. Again, choose $t_n \to -\infty$ such that $z_n := \gamma(t_n) \to z_0$; we may suppose that $t_{n+1} \leq t_n - 1$ for all

n. Let D be the closed hyperbolic disk

$$D := \{z \in U\colon\ \operatorname{dist}_U(z, z_0) \leq 2\ell_0\}.$$

Since h strictly expands the hyperbolic metric, there exists a number $\lambda > 1$ such that $\|\operatorname{D}h(z)\|_{\text{hyp}} \geq \lambda$ for all $z \in U$ with $h(z) \in D$; in fact, we can take $\lambda := \max_{z\in D} \frac{\rho_U(z)}{\rho_{h(U)}(z)}$ (where ρ_U and $\rho_{h(U)}$ are the densities of the corresponding hyperbolic metrics).

If n_0 is large enough, then $\operatorname{dist}_U(\gamma(t_n), z_0) < \ell_0$ for $n \geq n_0$, and thus $\gamma\big([t_n, t_n{+}1]\big) \subset D$. Therefore

$$\begin{aligned}\ell_{t_n} = \ell_U\big(\gamma[t_n, t_n + 1]\big) &= \ell_U\big(h(\gamma[t_n - 1, t_n])\big)\\ &\geq \lambda\ell_U\big(\gamma([t_n - 1, t_n])\big) = \lambda\ell_{t_n-1} \geq \lambda\ell_{t_{n+1}}.\end{aligned}$$

By induction, $\ell_{t_n} \leq \lambda^{n_0-n}\ell_{t_{n_0}}$ for $n \geq n_0$, which means that $\ell_t \to 0$ as $t \to -\infty$. Therefore

$$\operatorname{dist}_U(z, h(z)) = \lim \operatorname{dist}_U(z_n, h(z_n)) = \lim \ell_{t_n} = 0,$$

as required.

Finally, suppose that $h(U) = U$. Then by assumption and by [M, Theorem 5.2], the iterates of f tend locally uniformly to ∞ (in the sense that they leave every compact subset of U). , or we are in one of the exceptional cases from the statement of our theorem. In the former case, it follows easily that γ has no accumulation points in U at all.

If $\gamma|_{[0,1]}$ is not smooth, then the proof can easily be modified in a number of ways. For example, let us define $d(t_1, t_2)$ to be the (hyperbolic) length of the unique hyperbolic geodesic of U homotopic to $\gamma|_{[t_1,t_2]}$. Then we can set

$$\ell(t) := \max_{s\leq t} d(s, s + 1) = \max_{s\in[t-1,t]} d(s, s + 1),$$

and the proof goes through as before. (Alternatively, replace $\gamma|_{[0,1]}$ by a smooth curve $\tilde{\gamma}$ in the same homotopy class and continue $\tilde{\gamma}$ using pullbacks. Using hyperbolic contraction arguments as above, it is easy to see that both curves will have the same accumulation sets.) ■

Theorem B.2 (Fixed rays do not land at infinity)**.**
Let $f \in \mathcal{B}$, and suppose that $\gamma\colon (-\infty, 1] \to \mathbb{C}$ is a curve with $f(\gamma(t)) = \gamma(t + 1)$ for all $t \leq 0$. If $z_0 \in \hat{\mathbb{C}}$ with $\lim_{t\to-\infty} \gamma(t) = z_0$, then $z_0 \neq \infty$.

Proof. Suppose, by contradiction, that $z_0 = \infty$. Then γ is an extendable set, which is impossible by Corollary 3.5. ■

Corollary B.3 (Landing criterion)**.**
Let $f \in \mathcal{B}$, and let $\gamma\colon (-\infty, 1] \to I(f)$ with $f(\gamma(t)) = \gamma(t + 1)$. Then γ lands at a repelling or parabolic fixed point of f if and only if there exists some domain U such that $U \subset f(U)$, $f\colon U \to f(U)$ is a covering map and $\gamma\big((-\infty, T]\big) \subset U$ for some $T < 0$.

Proof. The "if" part is a combination of Theorem B.1, Theorem B.2 and the classical Snail Lemma. The "only if" part is trivial since we can let U be a linearizing neighborhood or repelling petal. ■

Corollary B.4 (Rays not intersecting the postsingular set)**.**
Let $f \in \mathcal{B}$ and γ as above, and suppose that $\gamma\big((-\infty,T]\big) \cap \mathcal{P}(f) = \emptyset$ for some $T < 0$. Then γ lands at a repelling or parabolic fixed point of f.

Proof. Let $V := \mathbb{C}\setminus\mathcal{P}(f)$, and let U be the component of $f^{-1}(V)$ containing $\gamma\big((-\infty,T-1]\big)$. The claim follows from the landing criterion. ∎

Corollary B.5 (Polynomial rays)**.**
Let f be a polynomial. Then every periodic ray of f lands at a repelling or parabolic fixed point of f.

Proof. Let U be the basin of infinity and apply Theorem B.1. ∎

Both Corollary B.3 and Corollary 5.3 give necessary and sufficient conditions for fixed rays to land. However, in practice it appears to be often difficult to obtain these conditions. Let us consider the family $E_\kappa\colon z \mapsto \exp(z) + \kappa$ of exponential maps as an example. Suppose that, for some E_κ, there exists a fixed ray γ which accumulates on the singular value κ, whose orbit in turn accumulates on all of γ. (It can easily happen that the singular orbit is dense in $\mathbb{C}$; in fact this behavior is generic in the bifurcation locus [R1, Theorem 5.1.6].)

It is easy to see that, in this situation, no set U as in Corollary B.3 could exist. Thus, in order to prove landing of γ by a hyperbolic contraction result, we need to *a priori* show that γ does not accumulate at κ. This would appear to be extremely difficult without any prior dynamical information: even in well-controlled cases there are dynamic rays which do not land, and whose accumulation sets are actually indecomposable continua [R5].

In the exponential family, this problem can be circumvented by using parameter space arguments. This gives some hope that more general landing theorems are also true in higher-dimensional parameter spaces such as the space of cosine maps, but unfortunately the methods in the exponential case break down completely. On the other hand, Corollary 5.3 shows that failure of univalence really is the only possible obstruction to landing of periodic rays.

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